Scientific Curiosity

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Abstract

Regardless of the geographic and of the historical context, the primary source of scientific inquiry has always been curiosity. Prior to any economic motivation, people engage on research activities because of their urge to know and of their willingness to discover novelties. Over time, curiosity has led the attention of those engaged in the quest for knowledge to be dispersed across all virtually imaginable subjects. In this paper, a theoretical model is devised to investigate the implications of the curiosity driven dispersion of attention. The main implication is that even under a scenario of constrained available attention, its dissemination across a progressively wider array of scientific objects is capable of conducting to an outcome of sustained long-term growth of the number of scientific endeavors undertaken with success.

Keywords: Curiosity; Science frontier; Dispersed attention; Bernoulli trials; Geometric distribution; Endogenous growth.

Título: Curiosidade Científica

Resumo: Independentemente do contexto geográfico ou histórico, a fonte primordial de pesquisa científica sempre foi a curiosidade. Antes de qualquer motivação económica, as pessoas envolvem-se em atividades de investigação devido ao seu impulso em conhecer e à sua determinação em descobrir coisas novas. Ao longo do tempo, a curiosidade levou a que a atenção dos cientistas se dispersasse virtualmente por todos os assuntos imagináveis. Neste artigo, um modelo teórico é desenvolvido com o objetivo de investigar as implicações da dispersão da atenção guiada pela curiosidade. A principal implicação é que mesmo num cenário de atenção limitada, a sua disseminação através de um número progressivamente maior de objetos científicos é capaz de conduzir a um resultado de crescimento sustentado de longo prazo do número de atividades científicas prosseguidas com sucesso.

Palavras-chave: Curiosidade; Fronteira da ciência; Dispersão de atenção; Tentativas de Bernoulli; Distribuição geométrica; Crescimento endógeno.

But curiosity does not mean and has never meant just a single thing. Even if we accept the modern definition of `eagerness to know or learn something', there are many ways to be curious. One can flit in gadfly manner from one question to another, acquiring little bits of knowledge without ever allowing them to cohere and mature into a real understanding of the world's mechanisms. One can store up snippets of information like a miser, never putting them to good use. One can pose questions idly or flippantly, with no plan for coherent enquiry into nature. One can be curious about matters that really are none of one's business, such as the sexual habits of one's neighbours. But one can also seek knowledge with serious and considered intent -- and may then do so either in the manner of Isaiah Berlin's fox who would know many little things, or as the hedgehog who knows a single thing profoundly. One can be curious obsessively, or passionately, or soberly, or with clinical detachment.

Ball (2012, p.7)

1. Introduction

From the tiniest insect to the largest galaxy in the universe, the domains of science extend across all the living beings, inert objects, social processes and abstract constructions. Scientific knowledge is concerned with everything that captures human attention and imagination.

Notwithstanding its pervasiveness, the matters of interest to science are constantly expanding due to the unlimited curiosity that we know to be innate to mankind. Even when one apparently has reached the limits of human capabilities to approach and act upon the surrounding world, new avenues for the exploration of reality open up to the inquisitive mind: the invention of the telescope and of the microscope allowed to continue to contemplate nature beyond the potential of human eyesight; increasingly efficient transportation, by land, sea or air, led to the enhancement of the possibilities to travel farther and faster; advances in medicine redounded on the discovery of ways to mitigate or eradicate many infirmities including highly contagious diseases.[1]

The above reasoning suggests that scientific knowledge knows no boundaries, hence being subject to sustained and persistent growth. But how is this possible in a world where the resources that might be allocated to research are inevitably scarce? In this paper, a framework of analysis is proposed to confront unbounded scientific curiosity with the limited amount of attention humanity can allocate to research activities. It turns out that the solution to this puzzle is related with the idea of dispersion: even in a setting where a fixed amount of attention is available, dispersing attention across a progressively wider array of scientific challenges results in perpetual growth. This result does not require an

[1] Technical and scientific advances, as the ones mentioned, are obviously the climax of processes where curiosity led to step-by-step incremental innovations that ultimately culminated in important improvements for the existence of human beings. The invention and first uses of the telescope and of the microscope by Hans Lippershey, Zacharias Janssen, Cornelis Drebbel, and Galileo Galilei, the development of aeronautics by Sir George Cayley, Otto Lilienthal, Horatio Phillips, and the Wright brothers, or the discovery of vaccines by Edward Jenner, Louis Pasteur, or Maurice Hilleman, are not just the straightforward outcome of the resilience of these brilliant minds; but it was their curiosity, building upon the curiosity of many that preceded them, that allowed science and knowledge to evolve in benefit of all mankind.

increasing value of new vintages of knowledge, neither a strong obsolescence of old knowledge, nor any other artifice.

In order to formally approach scientific curiosity, we introduce the notion of science frontier. The science frontier will emanate from a discrete-time distribution translating the probable number of experiments the scientific community has to undertake before achieving success on a given research project. These experiments take the designation of Bernoulli trials. A constant stock of attention is considered, and a choice has to be made by the scientific community: to concentrate attention over time in the scientific puzzles that remain to be solved or, alternatively, to disseminate attention by an ever increasing number of new research challenges.[2]

Attention dissemination is the option leading to sustained growth, as the analysis will reveal. Attention dissemination is also the possibility that best fits the above arguments on the role of curiosity: the willingness to explore new physical, social or conceptual worlds, searching for new knowledge, is the force that will shape new scientific frontiers, thus triggering a mechanism of perpetual scientific self-regeneration.

The contents of the paper proceed as follows. Section 2 briefly discusses the role of curiosity in expanding science and the universe of scientific objects. Section 3 describes the notion of science frontier and the dynamics underlying the emergence of new research questions. Section 4 is dedicated to the analysis and discussion of long-term growth. In section 5, a few potential extensions of the model are put into perspective. Section 6 concludes.

2. A Note on Curiosity and the Scientific Inquiry

From the writings of the psychologist Robert Plutchik (Plutchik, 1980), one draws two important ideas about curiosity. First, that it can be classified as an emotion indelibly associated with inquisitive thinking and investigative behavior; second, that it is conceivable as a feeling arising in the confluence of two primary emotions: trust and surprise. This and the common sense perception one has about curiosity, make it evident the difficulty in classifying such entity: is it an emotion, a feeling, a quality, a flaw, a behavioral attitude, a spontaneous reaction to external stimuli? In fact, curiosity has a multifaceted nature, and it can be any of these, depending on the perspective from which it is being looked upon.

In order to launch the discussion on the role of curiosity as a driver of scientific progress, we briefly highlight a few of its most prominent features, mostly agreeing with the reflection of Loewenstein (1994), and Golman and Loewenstein (2015) on the subject: (*i*) A plain definition of curiosity is that it is a compulsion to know, a willingness to obtain information for its own sake, regardless of the underlying utility it might bring. Hence, from the point of view of rational choice, curiosity might be interpreted as an anomaly; (*ii*) Curiosity is transient, volatile and superficial, since it can be ignited suddenly and with no apparent reason, it can change its nature fast, and it eventually disappears at a

[2] In conceptual terms, the adopted notion of science frontier has some parallel with Kortum's (1997) technology frontier, a popular analytical and geometrical tool used by economists to address issues pertaining to international trade and growth.

glance; (*iii*) There is not a clear cut transparent relationship between curiosity and knowledge. Curiosity motivates the quest for knowledge and the urgency in solving problems. But new discoveries might also unveil hidden information that exerts a negative influence over the subsequent willingness to know; (*iv*) Curiosity is attached with the human tendency to fill in information gaps. When people realize the existence of an information gap, they will search for the associated knowledge, even when the expected outcome is not likely to bring any utility gain; (*v*) Technical studies on curiosity typically associate it with the notion of entropy (Kim *et al.*, 2013). By collecting information, individuals satisfy curiosity, learn about random outcomes or uncertain beliefs, and consequently reduce entropy.

Confining the discussion to the domain of scientific research, one finds curiosity in the short list of motives underlying the choice of the subjects scientists express interest in approaching. In Lam (2011), three meaningful motives for scientists to engage in research activities are identified, which receive the nicknames `gold', `ribbon' and `puzzle'. The gold motive concerns financial rewards, the ribbon motive is attached to peer recognition and reputation gains, and the puzzle motive is related with the intrinsic satisfaction that comes from investigating and discovering new things. It should be evident that this third motive, puzzle, is the one directly associated with intellectual curiosity. Although a significant portion of the scientific activity in modern days is pursued within formally organized environments (universities, laboratories, firms,), making material gains and reputation important elements of the research activity, science is, in fact, more than this. Inside and outside the mentioned environments, there is a natural proclivity for human beings to be creative, to search for new knowledge and to fill in their information gaps. The puzzle motive for research is the most pervasive of the three mentioned, and it is the one that most easily thrives in any setting, independently of the degree of organization of purposive research activities.

Along the same lines, Alon (2009) emphasizes that the emergence of scientific endeavors is shaped by the weighting of two main dimensions, along which the researcher has to equate what a meaningful research project truly is. One of the dimensions is feasibility, i.e., the extent in which a problem is easy or difficult to approach. The other dimension is named interest, and it translates the knowledge one expects to attain when addressing such issue. Clearly, in such a perspective, interest is directly associated with the desire to fill in information gaps, and therefore it can be renamed as curiosity. In this context, feasibility and interest might be conceived as the two axes of a two-dimensional space containing a Pareto frontier. Problems located in the frontier represent the optimal pairs feasibility-interest and, thus, those that scientists should address (problems beyond the frontier are unapproachable under the current state of knowledge; those between the axes and the frontier are irrelevant or already solved). The choice of a problem in the Pareto frontier should then weight the curiosity it generates against the expected outcome in terms of feasibility. Although curiosity is important as a criterion to select scientific challenges, it must be balanced with the reasonability or feasibility of the research goals.

The history of scientific accomplishments and advancements exposes two general important features: first, science is cumulative, it deconstructs prior knowledge at the same time it recognizes its value and incorporates it in the new paradigms; second, science tends towards dispersion, in the sense that new discoveries typically activate a wide array of new scientific challenges that researchers become interested in approaching. The second of these features, dispersion, although less salient, has been of key importance for the progress of human knowledge. As the reflection on the evolution of science in Renaissance Europe by Ball (2012) thoroughly describes, science ended up by being interested in everything, from the exotic animals and plants of the new world to the stars and planets in the outer space, to the observation of physical phenomena and to chemistry experiments, or to the fine analysis of microcosms.

When all the questions and doubts one can imagine are likely to emerge in the mind of scientists, we find no boundaries to what science can approach and achieve and we will have an endless process of systematic inquiry about everything we encounter in nature, in society and in the laboratory. Even recognizing that the resources one can allocate to research are necessarily bounded (being attention the fundamental resource in this respect), curiosity will have no bounds, and thus the stock of attention will disseminate to an ever increasing number of scientific problems. This confrontation between limited resources and the urge to spread them in order to tackle with more and more interrogations is the subject that motivates the model in the following sections and the result that sustained growth is possible even in a world of limited resources.

The pervasive nature of science is also, in any case, supported by the technical progress it fosters. As science evolved along the last few decades, powerful and unprecedented computational resources have emerged, allowing for scientific knowledge and scientific curiosity to expand even further. Therefore, the model to develop has also an underlying technical and computational essence: scientific curiosity promotes the creation of progressively more powerful computational techniques. These powerful tools allow, in turn, for scientific curiosity to reach areas of knowledge of increasingly difficult access for the human mind and areas one could not imagine to be possible to approach in previous stages of scientific development. As computational tools become further and further sophisticated curiosity will necessarily follow the evolution of techniques.

3. Research Dynamics

3.1 The Science Frontier

Let $\mathbf{Q}(t)$ be the finite set of scientifically meaningful unsolved or open questions that the research community is aware of at time period $t = t_0, t_0 + 1, \ldots$ A question $Q(t_0 + T, t_0) \in \mathbf{Q}(t)$ is a scientific problem formulated at date $t = t_0$ and under scrutiny at period $t = t_0 + T$, $T = 0, 1, \ldots$ At $t = t_0$, an at most countable set of mutually exclusive possible answers $\mathbf{A}(t_0, t_0)$ exists associated with each question $Q(t_0, t_0)$. Scientists will search for the single element of $\mathbf{A}(t_0, t_0)$ that delivers the right answer to the research question. The probability of discovering the correct answer when undertaking a scientific experiment at $t = t_0$, which will be designated as *probability of success*, is denoted by $p(t_0, t_0)$. The *probability of failure* is, thus, $q(t_0, t_0) = 1 - p(t_0, t_0)$.

At the formulation date, researchers will have the opportunity of undertaking an unspecified number of experimentation rounds $k \in \mathbb{Z}^+$. These research rounds are Bernoulli trials, i.e., they are modeled as a sequence of independent trials of an experiment that has exactly two possible outcomes (success and failure), and where the probability of success is the same for every trial.[3] In this context, a geometric distribution is brought into the analysis; this is a discrete distribution that indicates the number of experiments that are necessary to obtain the first success. The cumulative distribution function (cdf) of the geometric distribution, which represents the probability of obtaining a successful result in trial *k* or in a trial previous to *k*, is $F(k,q(t_0,t_0)) = 1 - q(t_0,t_0)^k$.

The science frontier will be defined, in this environment, as the probability of not having achieved success after *k* trials, i.e.,

$$G(k,q(t_0,t_0)) = 1 - F(k,q(t_0,t_0)) = q(t_0,t_0)^k \tag{1}$$

Assuming a large number of scientific questions formulated or activated at $t = t_0$, $G(k,q(t_0,t_0))$ has a second interpretation; it also stands for the percentage of scientific questions that remain still unsolved after *k* rounds of experimentation. Fig.1 portrays the science frontier for three different values of the probability of success per experiment. The figure allows to visualize the intuitive result that a lower probability of failure (a higher probability of success) pulls the frontier downward, meaning that more scientific questions receive a correct answer after some number of experimentation rounds or Bernoulli trials.

[3] Modelling research rounds as Bernoulli trials implies that each scientific experiment is independent of the previous one. Obviously, this is a strong assumption that overlooks incremental learning that may come from failed trials; after all, in science, as in many other aspects of life, mistakes and failures often lead us, sooner or later to the right answer. In other circumstances, wrong solutions may deviate the researcher from success in a way that compromises further experiences. The independence assumption directs us, implicitly, to a convenient offsetting scenario: wrong answers produce some learning but also have a dismantling effect that together, in our framework, make the subsequent experience not to depend neither positively or negatively from the one that preceded it.

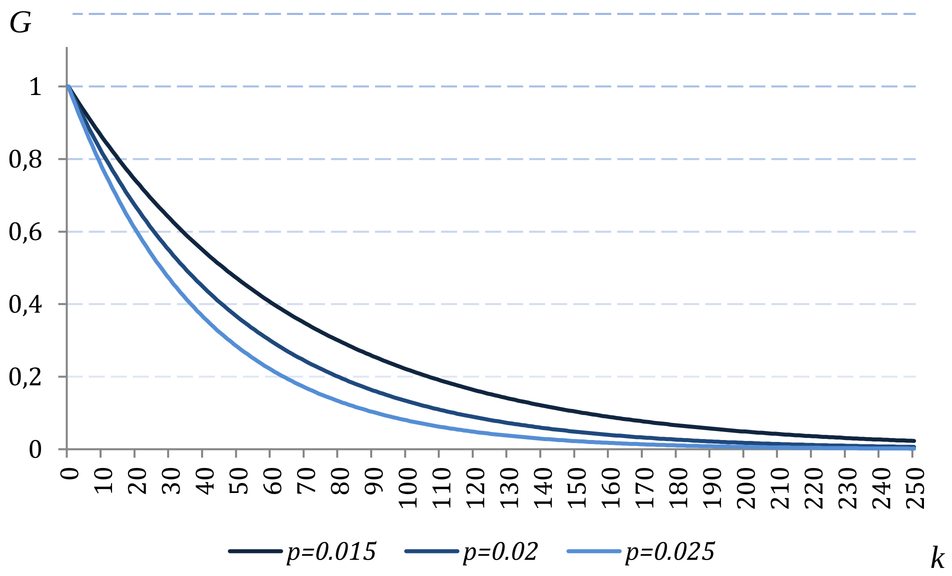

Fig.1. Science frontier ($p = 0.015$, $p = 0.020$, $p = 0.025$).

3.2 Moving Frontiers

At the beginning of period $t = t_0 + 1$, a share $1 - G(k, q(t_0, t_0))$ of scientific challenges raised at $t = t_0$ has been solved, but a percentage $G(k, q(t_0, t_0))$ remains to be resolved. The geometric distribution is memoryless, what signifies that if no change is observed on the probability of success from one period to the next, then researchers will be confronted with exactly the same science frontier in period $t = t_0 + 1$ relatively to the one faced at $t = t_0$. However, one might conceive a scenario where the probability of success changes as the result of a variation in the level of the attention given to each scientific problem. Let $w(t_0, t_0) > 0$, a given value, be the attention attributed to a question $Q(t_0, t_0)$ and let $\Delta w(t_0, t_0)$ be the change on attention from period $t = t_0$ to period $t = t_0 + 1$. The variation in attention is assumed to occur, in the proposed setting, at constant rate $\delta \in \mathbb{R}$,[4]

[4] The exogeneity and constancy over time of parameter δ is an analytically convenient simplifying assumption that deserves some reflection. By assuming a constant δ, one considers a mechanical process through which scientific challenges gain ($\delta > 0$), maintain ($\delta = 0$) or lose ($\delta < 0$) relevance in the mind of scientists. The reasons why one of these options will prevail and persist over the others can be of various natures. Scientific problems might be perceived as stimulating challenges that researchers desire to explore further and further or, in opposition, the systematic failure in finding convincing solutions for research questions may lead researchers to progressively abandon such questions in favor of other challenges (e.g., if systematic attempts to travel to Mars are failed, the attention of scientists and engineers may relocate elsewhere).

$$\Delta w(t_0, t_0) = \delta \times w(t_0, t_0) \tag{2}$$

The probability of failure at $t = t_0 + 1$ will be defined as $q(t_0, t_0)$ multiplied by a term representing the fall (increase) in the probability of failure when attention increases (falls), i.e.,

$$q(t_0 + 1, t_0) = q(t_0, t_0) \times q(t_0, t_0)^{\Delta w(t_0, t_0)} \tag{3}$$

Given the cdf, the same kind of behavior will characterize the motion of the science frontier,

$$G(k, q(t_0 + 1, t_0)) = G(k, q(t_0, t_0))^{1+\Delta w(t_0, t_0)} \tag{4}$$

Equation (4) indicates that the science frontier may remain unchanged from one period to the next, when $\delta = 0$; for $\delta > 0$, the increase in attention will make the science frontier to contract towards the origin, while $\delta < 0$ represents a scenario of decreasing attention, implying an outward shift in the science frontier G.

At the end of period $t = t_0 + 1$, after k Bernoulli trials are again implemented, one knows that a share $1 - G(k, q(t_0 + 1, t_0))$ of the questions needing answer at $t = t_0 + 1$ have been effectively answered in this period, while the remaining share, $G(k, q(t_0 + 1, t_0))$, requires eventual further investigation in the next period. Therefore, if $Q(t_0)$ represents the number of activated questions at $t = t_0$, then, after $t = t_0 + 1$, the number of already solved scientific mysteries is

$$\begin{aligned} A(t_0 + 1) &= Q(t_0) \times \{[1 - G(k, q(t_0, t_0))] + G(k, q(t_0, t_0)) \times [1 - G(k, q(t_0 + 1, t_0))]\} \\ &= Q(t_0) \times [1 - G(k, q(t_0, t_0)) \times G(k, q(t_0 + 1, t_0))] \end{aligned} \tag{5}$$

The number of questions still waiting for a successful response amounts to

$$Q(t_0) - A(t_0 + 1) = Q(t_0) \times G(k, q(t_0, t_0)) \times G(k, q(t_0 + 1, t_0)) \tag{6}$$

The above reasoning applies as well to all the subsequent time periods. Thus, one considers the following general rules for the evolution of attention and for the motion of the science frontier,

$$\Delta w(t_0+t+1,t_0)=\delta w(t_0+t,t_0),\, w(t_0,t_0) \text{ given} \tag{7}$$

$$G(k,q(t_0+t+1,t_0)) = G(k,q(t_0+t,t_0))^{1+\Delta w(t_0+t,t_0)},\, w(t_0,t_0),\, q(t_0,t_0) \text{ given} \tag{8}$$

Through recursive substitution, it is straightforward to obtain the solution of the difference equation (8), which might be displayed under the form

$$G(k,q(t_0+T,t_0)) = G(k,q(t_0,t_0))^{\exp\left[\sum_{\tau=0}^{T-1}\ln(a_\tau)\right]},\, T>0 \tag{9}$$

with $a_\tau \equiv 1+\delta(1+\delta)^\tau w(t_0,t_0)$.

Equation (9) represents the share of questions raised at moment $t=t_0$ for which a correct solution has not yet been discovered after T periods of time. From this solution, one extracts the following result,

Proposition 1. *Regardless of the sign of the parameter representing the growth rate of attention (δ), the number of questions that are activated at period $t=t_0$ and that remain to be solved in the long-term converges to zero.*

Proof See appendix ■

Under an infinite horizon, in the proposed setting, every scientific challenge posed to researchers at some initial period will end up by getting an answer, what is a direct consequence of the attention that continues to be given to the questions period after period until a solution is met, independently of the upward or downward trajectory followed by the allocated attention (i.e., as long as some focus is put into a research challenge, there will always be a positive probability of success). Obviously, the convergence towards the steady-state of zero unsolved problems will be faster when the attention is increasing than in the opposite circumstance. Fig. 2 illustrates this reasoning.

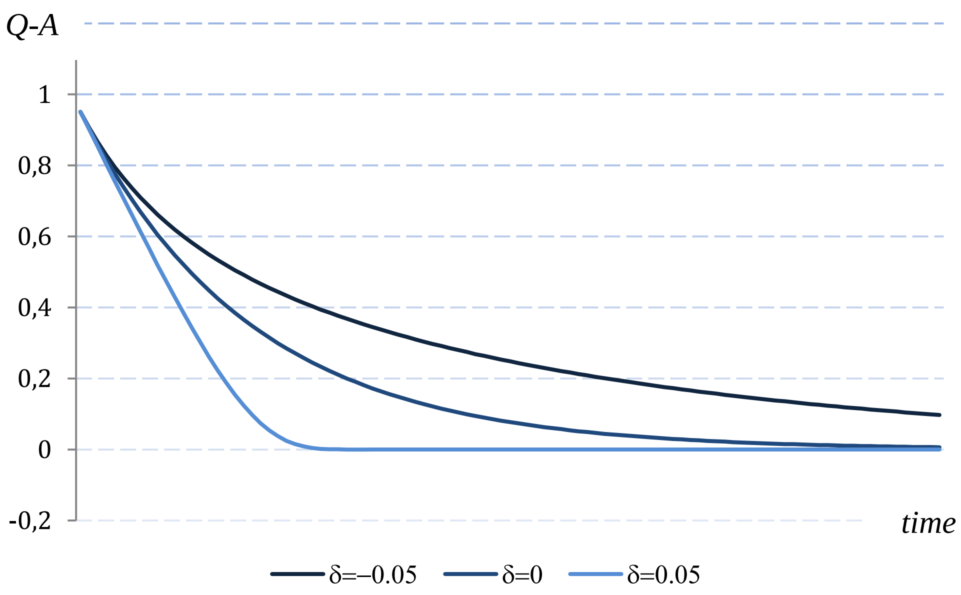

Fig.2. Trajectories of unanswered questions

$(\delta = -0.05,\ \delta = 0,\ \delta = 0.05;\ k = 5;\ w(t_0,t_0) = 1;\ q(t_0,t_0) = 0.99;\ Q(t_0) = 1)$.

In the setting under assessment, the number of answered questions at date $t = t_0 + T$ is presentable as

$$A(t_0 + T) = Q(t_0) \times \left(1 - G(k, q(t_0,t_0))^{1+\sum_{\zeta=0}^{T-1}\left\{\exp\left[\sum_{\tau=0}^{\zeta} \ln(a_\tau)\right]\right\}}\right) \tag{10}$$

Evidently, $\lim_{T\to\infty} A(t_0 + T) = Q(t_0), \forall\delta \in \mathbb{R}$.

3.3 Vintages of Scientific Challenges

In the previous subsection, the dynamics of science was approached taking into account solely the questions activated at the initial date. In practice, new research challenges are likely to emerge in the minds of researchers, driven by curiosity, with a systematic cadency, implying that new scientific problems might presumably erupt at every date from $t = t_0$ to the assumed time horizon.

To address the systematic emergence of new questions start by assuming the availability of a constant stock of attention: $W > 0, \forall t$. The constant stock of attention assumption is an essential element of the analysis. Although the attention attributed to scientific issues has historically grown over time, our argument in this context is that it can be interpreted as an exhaustible resource: there are limits to the attention that can be allocated to research questions, both in terms of the number of individuals assigned to scientific quests and in terms of the time and effort each person, individually, can associate to the process of discovery. An alternative hypothesis, perhaps more realistic, would be to assume an increasing stock of attention subject to diminishing returns; however, this will not produce any significant change regarding the long-term results to be obtained. The assumption of a constant stock of attention is later relaxed in order to compare the implications of both scenarios (without and with growth of the attention endowment).

Given the stock of attention, and the attention required to seize each question at $t = t_0$, $w(t_0, t_0)$, the number of activated questions at the initial period will be endogenously determined: $Q(t_0) = \frac{W}{w(t_0, t_0)}$. The scientific community is able of solving $A(t_0) = Q(t_0) \times [1 - G(k, q(t_0, t_0))]$ questions, potentially freeing a part of the available attention to new challenges that may emerge in the next period.

At $t = t_0 + 1$, $Q(t_0) \times G(k, q(t_0, t_0))$ questions of the first vintage remain to be answered. Let the attention needed to answer all questions transferred from $t = t_0$ to $t = t_0 + 1$ be represented by $W(t_0 + 1, t_0)$. This variable will correspond to the number of questions requiring attention after not being solved at $t = t_0$ times the attention that each one will receive at $t = t_0 + 1$, i.e., $w(t_0 + 1, t_0) = (1 + \delta) \times w(t_0, t_0)$,

$$W(t_0 + 1, t_0) = [Q(t_0) \times G(k, q(t_0, t_0))] \times w(t_0 + 1, t_0) \tag{11}$$

The evaluation of the allocation of attention on the second date requires separating the different cases arising from the different signs parameter δ may take. In this subsection, we briefly look at the simplest case, i.e., $\delta = 0$, and leave the cases of concentrated attention and dispersed attention to the next subsection.

Under $\delta = 0$, the attention required to approach at $t = t_0 + 1$ the scientific questions raised at $t = t_0$, is the product between the stock of attention and the amount of questions that remain to be successfully answered, i.e., $W(t_0 + 1, t_0) = W \times G(k, q(t_0, t_0))$. Consequently, a level of attention $W(t_0 + 1, t_0 + 1) = W \times [1 - G(k, q(t_0, t_0))]$ is now available for the introduction of new questions. The number of new interrogations that is activated will be $Q(t_0 + 1, t_0 + 1) = Q(t_0) \times [1 - G(k, q(t_0, t_0))]$. As it should be evident, the number of challenges that will be available to explore at $t = t_0 + 1$ is identical to the respective initial number,

$$Q(t_0 + 1) = Q(t_0) \times G(k, q(t_0, t_0)) + Q(t_0 + 1, t_0 + 1) = Q(t_0) \tag{12}$$

In this straightforward setting, where no distinctive feature separates questions activated at one period or the next, the number of questions to address at each time period will be exactly the same. As a consequence, science will asymptotically stop growing in the long-term; the reason is that at each period a same stock of questions is solved, $A(t) = Q(t_0) \times [1 - G(k, q(t_0, t_0))]$, leading to an accumulated number of solved questions of $\widehat{A}(t_0 + T) = (T+1) \times Q(t_0) \times [1 - G(k, q(t_0, t_0))]$. The growth rate of $\widehat{A}(T)$ is

$$\gamma_{\widehat{A}} = \frac{1}{T+1}, \tag{13}$$

a value that converges to zero as T goes to infinity.

3.4 Concentrated Attention vs Dispersed Attention

Although straightforward to analyze, case $\delta = 0$ comprises no noteworthy outcome: in each period, a given quantity of science challenges is solved, freeing resources to approach other challenges that receive exactly the same attention as those that were successfully approached, what extinguishes, in the long-run, the ability of the scientific knowledge to continue growing. In this subsection, we look at the alternative settings, namely the ones in which the scientific community, when confronted with unsuccessful research experiments, increases the attention allocated to already existing questions ($\delta > 0$) or, alternatively, decreases it ($\delta < 0$).

Consider case $\delta > 0$. In this case, equation (11) encloses three different situations, namely the following:

1) The attention available is equal to the attention required to approach the questions that were passed on from the first period to the next, $W = W(t_0 + 1, t_0)$. This scenario occurs when condition $G(k, q(t_0, t_0)) \times (1 + \delta) = 1$ is satisfied. All questions ignited at $t = t_0$, which have not received a successful answer in that period, are again addressed at $t = t_0 + 1$, but no attention is left to confront any other challenges. Analytically,

$$Q(t_0 + 1, t_0) = Q(t_0) \times G(k, q(t_0, t_0)) = \frac{Q(t_0)}{1 + \delta} \tag{14}$$

2) The available stock of attention falls short of the required attention to address unanswered questions of the first vintage, $W < W(t_0 + 1, t_0)$. This condition is equivalent to $G(k, q(t_0, t_0)) \times (1 + \delta) > 1$. Now, some of the questions activated at $t = t_0$ and that have not been successfully answered in the initial date, have to be discarded. The number of questions activated at $t = t_0$ that will be addressed at $t = t_0 + 1$ is derived from constraint

$$W = Q(t_0 + 1, t_0) \times (1 + \delta) \times w(t_0, t_0) \tag{15}$$

Rearranging and recalling that $W = Q(t_0) \times w(t_0, t_0)$,

$$Q(t_0 + 1, t_0) = \frac{Q(t_0)}{1 + \delta} \tag{16}$$

It is straightforward to confirm that, in this setting, $Q(t_0 + 1, t_0) < Q(t_0) \times G(k, q(t_0, t_0))$, since this is equivalent to $\frac{1}{1+\delta} < G(k, q(t_0, t_0))$, which is the assumption underlying the current situation. The number of discarded questions initially activated is easy to identify:

$$Q(t_0) \times G(k, q(t_0, t_0)) - Q(t_0 + 1, t_0) = \frac{Q(t_0)}{1 + \delta}[G(k, q(t_0, t_0)) \times (1 + \delta) - 1] \tag{17}$$

which is a positive value. Note also, in this case as in the previous one, that $Q(t_0 + 1, t_0 + 1) = 0$.

3) The available stock of attention exceeds the required attention to address all questions yet to answer from vintage $t = t_0$, i.e., $W > W(t_0 + 1, t_0)$. Now, $G(k, q(t_0, t_0)) \times (1 + \delta) < 1$ and researchers will be not only able to fully address the set of questions that have passed from $t = t_0$ to $t = t_0 + 1$, but novel questions can be activated. The relevant constraint is, in this circumstance,

$$W = Q(t_0) \times G(k, q(t_0, t_0)) \times (1 + \delta) \times w(t_0, t_0) + Q(t_0 + 1, t_0 + 1) \times w(t_0 + 1, t_0 + 1) \tag{18}$$

Assuming that new questions receive exactly the same attention as the ones from the previous vintage, i.e., $w(t_0 + 1, t_0 + 1) = w(t_0 + 1, t_0) = (1 + \delta) \times w(t_0, t_0)$, and solving the equation with respect to $Q(t_0 + 1, t_0 + 1)$,

$$Q(t_0 + 1, t_0 + 1) = Q(t_0)\left[\frac{1}{1 + \delta} - G(k, q(t_0, t_0))\right] \tag{19}$$

The total number of questions being addressed after the end of period $t = t_0 + 1$ is

$$Q(t_0 + 1, t_0) + Q(t_0 + 1, t_0 + 1) = \frac{Q(t_0)}{1 + \delta} \tag{20}$$

as in the other two scenarios.

The above analysis might be continued for $t = t_0 + 2$, and then extrapolated for subsequent periods of time, as well as for case $\delta < 0$. The main outcome emerging from such computation is that independently of the relation between available and required attention and independently of the sign of δ, the number of activated questions will be, at every future date $t = t_0 + T$,[5]

$$Q(t_0 + T) = \frac{Q(t_0)}{(1 + \delta)^T} \tag{21}$$

suggesting that in each period the following number of scientific challenges is successfully approached,

$$A(t_0 + T) = \frac{Q(t_0)}{(1 + \delta)^T} \times [1 - G(k, q(t_0 + T, t_0))] \tag{22}$$

The accumulated number of answered questions, in turn, is

$$\widehat{A}(t_0 + T) = Q(t_0) \times \sum_{t=0}^{T} \frac{1 - G(k, q(t_0 + t, t_0))}{(1 + \delta)^t} \tag{23}$$

To assess the long-term progress of scientific knowledge one evaluates, in the following section, the growth rate of the accumulated number of scientific challenges successfully answered, as depicted in (23).

[5] For reasons of space and fluidity of exposition, we refrain from presenting the complete derivation. This will be provided by the author to the interested readers upon request.

4. Balanced Growth

4.1 Dispersed Attention and Sustained Growth

The characterization of the dynamics of the science frontier has allowed to calculate the number of scientific challenges that, at period $t = t_0 + T$, are considered successfully solved by the scientific community. In the absence of knowledge obsolescence, such number corresponds to value $\widehat{A}(t_0 + T)$ in (23). As emphasized, this state of science is given by the same expression regardless of how the attention attributed to questions evolves over time, i.e., regardless of the sign of parameter δ.

However, it is precisely the sign of δ that will be decisive to determine how science grows in the long-run. In particular, it is straightforward to prove that concentrating attention or maintaining it at the same level over time ($\delta \geq 0$) will lead to a long-term steady-state of zero growth, while dispersing attention ($\delta < 0$) generates a balanced growth path (BGP) of sustained growth for science, i.e., the number of successfully approached scientific challenges will grow, after the transient phase as been surpassed, at a constant rate over time.[6]

We associate scenario $\delta < 0$ to scientific curiosity, in the sense that curious researchers will disperse their attention to progressively more subjects of analysis, and this works, without the need for any auxiliary processes, as an engine of persistent positive growth. On the contrary, if researchers further insist in trying to answer the same old questions that were not yet capable of providing a successful solution, this will be synonymous of growth exhaustion, when a fixed endowment of attention is taken.

To obtain the above characterized outcome, one needs to compute the growth rate of $\widehat{A}(t_0 + T)$ and to evaluate it when $T \to \infty$.

Proposition 2. *Let the state of science correspond, at a given date, to the so far accumulated level of successfully approached scientific challenges. For $\delta \geq 0$, in the long-term the state of science converges to a constant level; for $\delta < 0$, sustained growth is evidenced, i.e., the long-run growth rate of the state of science is a constant positive value.*

Proof See appendix ■

[6] This result is the corollary of our 'blind' intertemporal distribution of attention, which does not consider any explicit consequences of how attention is allocated. For instance, it is reasonable to infer that the successive escape forward implies costs, regarding knowledge accumulation, which probably have a negative impact on the ability to solve future problems. Those costs could be equated by bringing an additional parameter to the analysis. In such case, the eventuality of growth would be determined by the net effect of attention dispersion (the gain in approaching new challenges against the loss of relocating attention from open questions.

In Fig. 3, the three circumstances are depicted. Specifically, by considering $\delta = 0.05$, $\delta = 0$, and $\delta = -0.05$, it is visible that the first two cases imply time trajectories of $\hat{A}$ that converge to zero, while dispersed attention triggers convergence to a positive value; this value is derived in the proof of proposition 2 and corresponds to

$$\gamma_{\hat{A}} = -\frac{\Delta w(t_0 + T, t_0)}{w(t_0 + T + 1, t_0)} = -\frac{\delta}{1+\delta} \tag{24}$$

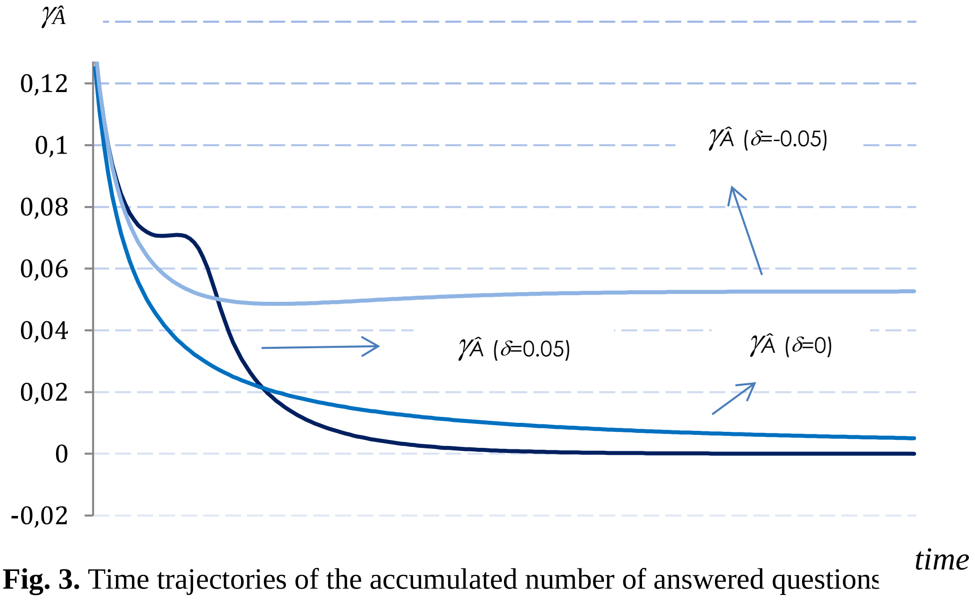

Fig. 3. Time trajectories of the accumulated number of answered questions
($\delta = -0.05$, $\delta = 0$, $\delta = 0.05$; $k = 5$; $w(t_0, t_0) = 1$; $q(t_0, t_0) = 0.99$; $Q(t_0) = 1$).

The figure makes it visible how scientific curiosity, interpreted as the interest for everything and the quest for never ending new challenges to human understanding, attributes to science the ability to reinvent itself over time. In a growth perspective, the figure highlights the potential of the science frontier model in accounting for growth diversity: a same framework is capable of generating zero long-term growth and positive constant long-term growth in a fixed endowment economy. Furthermore, by observing the graph, one realizes that in this specific case a tension exists between short-run and long-run growth: while dispersing attention is the only option for attaining a BGP of positive growth, in the initial periods science grows more when attention is concentrated; this might provoke a confrontation between the ultimate goal of expanding scientific knowledge in a perpetual basis and the immediate rewards that may come from a faster growth in the short-run.

4.2 Knowledge Obsolescence, Growing Attention and the Value of Innovation

The result for the BGP of scientific knowledge is potentially changed once we consider some incremental assumptions. Three of such meaningful assumptions are explored in this subsection for the relevant endogenous growth case, $\delta < 0$.

First, consider knowledge obsolescence, such that any discovery made at period $t = t_0 + T$ worth less at period $t = t_0 + T + 1$. Let $\sigma \in (0, 1)$ be the rate of obsolescence; at the end of period $t = t_0 + 1$, the value of the already solved questions is:

$$\widehat{V}(t_0 + 1) = Q(t_0) \times \left[(1 - \sigma) \times G(k, q(t_0, t_0)) + \frac{1}{1 + \delta} \times G(k, q(t_0 + 1, t_0)) \right] \tag{25}$$

And, expanding for the whole set of time periods,

$$\widehat{V}(t_0 + T) = Q(t_0) \times \sum_{t=0}^{T} \left[(1 - \sigma)^{T-t} \times \frac{1 - G(k, q(t_0 + t, t_0))}{(1 + \delta)^t} \right] \tag{26}$$

Second, abandon the constant attention endowment assumption and consider that the stock of attention grows at a constant positive rate, γ: $W(t + 1) = (1 + \gamma)W(t)$. The increasing available attention will allow to answer additional questions as time unfolds; for instance, in $t = t_0 + 1$, the constraint on the number of new activated problems will be

$$(1 + \gamma)W = [Q(t_0) \times G(k, q(t_0, t_0)) + Q(t_0 + 1, t_0 + 1)] \times (1 + \delta) \times w(t_0, t_0) \tag{27}$$

meaning that

$$Q(t_0 + 1, t_0 + 1) = Q(t_0) \times \left[\frac{1 + \gamma}{1 + \delta} - G(k, q(t_0, t_0)) \right] \tag{28}$$

and that

$$Q(t_0) \times G(k, q(t_0, t_0)) + Q(t_0 + 1, t_0 + 1) = \frac{1 + \gamma}{1 + \delta} \times Q(t_0) \tag{29}$$

Extrapolating for multiple time periods,

$$Q(t) = \left(\frac{1+\gamma}{1+\delta}\right)^t \times Q(t_0) \tag{30}$$

and the number of answered questions accumulated at period $t = t_0 + T$ will be

$$\widehat{V}(t_0 + T) = Q(t_0) \times \sum_{t=0}^{T} \left\{ \left(\frac{1+\gamma}{1+\delta}\right)^t \times [1 - G(k, q(t_0 + t, t_0))] \right\} \tag{31}$$

Finally, consider that questions from a newer vintage are more valuable than questions of older vintages. Let the additional value, from a vintage to the next, be given by parameter $v > 0$. This implies that, from $t = t_0$ to $t = t_0 + 1$,

$$\widehat{V}(t_0 + 1) = Q(t_0) \times \left[G(k, q(t_0, t_0)) + \frac{1+v}{1+\delta} \times G(k, q(t_0 + 1, t_0)) \right] \tag{32}$$

and, for $t = t_0 + T$,

$$\widehat{V}(t_0 + T) = Q(t_0) \times \sum_{t=0}^{T} \left\{ \left(\frac{1+v}{1+\delta}\right)^t \times [1 - G(k, q(t_0 + t, t_0))] \right\} \tag{33}$$

Equations (26), (31) and (33) all supply the value of scientific knowledge at point in time $t = t_0 + T$, with the value of science measured with respect to all current and previous scientific challenges successfully approached; BGP results are as follows,

Proposition 3. *Assume* $\delta < 0$. *In the presence of knowledge obsolescence, increasing attention and increasing value of new problem vintages, the long-term growth rate of scientific knowledge in the science frontier dynamic model becomes*

$$\gamma_{\widehat{V}} = \frac{(1+\gamma)(1+v)}{1+\delta} - 1 \tag{34}$$

Proof See appendix ■

Result (34) is a generalization of the growth rate already derived in the dispersed attention case. In particular, in the absence of attention growth ($\gamma = 0$) and increasing value of innovation ($\nu = 0$), we are back at growth rate (24). Thus, an immediate corollary of the result in proposition 3 is that knowledge obsolescence exerts no effect on long-run growth; independently of how fast old scientific knowledge loses value, this does not temper with how science, translated in the value of correctly answered questions, continues to accumulate in the long-term.

Therefore, the value of science grows over time, after the transient phase has been overcome, at a positive rate that depends on three entities: the rate of attention dispersion, the rate of change of the stock of attention, and the growth rate of the value of solved problems given the time period in which they are activated. Although the last two are obvious candidates to perpetuate growth, our main point is that they are not indispensable to achieve this result, since the dispersion of attention fulfills, *per se*, the same objective.

4.3 Entropy and Curiosity

Technical literature on curiosity typically establishes a bridge between curiosity and entropy. This is, explicitly, the case of the study by Golman and Loewenstein (2015), who design a utility function that contemplates both preferences over material gains and preferences over the value of information. In this specification of utility, the entropy of the probability distribution over possible answers enters as an argument of the function. The existence of many probable answers signifies a large level of entropy, while, on the other extreme, zero entropy is synonymous of complete certainty concerning the correct answer for a given problem. The preference of the agent will be for clarity, i.e., for less uncertain possible outcomes; therefore, a low value of entropy will benefit the agent in terms of her expected utility, implying that entropy arises in the utility function preceded by a minus sign.

The above reasoning suggests that curiosity and entropy are siamese twins. Curiosity does not exist in a state of zero entropy; however, it may acquire pathological levels when the certainty about the probable outcome is close to zero and, thus, entropy is very high. Furthermore, curiosity arises as a natural mechanism to counteract uncertainty, i.e., curiosity is a healing device against entropy.

The close link between curiosity and entropy applies, in the context of research dynamics, roughly in the same way as in any other context. Scientists search for solutions for activated problems, trying to fill in the perceived information gaps, in order to lower entropy. However, the final goal, as we have discussed in precedent sections, is not to reach a state of global zero entropy: scientists do not solve problems with the intent of attaining the definitive answers that would lead science to its final undisputable state. On the contrary, the curiosity of researchers leads them to encounter solutions to problems in order to be possible, afterwards, to formulate and approach other new enigmas. Therefore, while the objective of answering a research question is indeed to suppress the entropy associated with it, the ulterior goal is to open up the possibility for new questions to emerge. Because individuals are curious by nature, as the entropy associated with some class of problems eventually falls, new avenues to address novel highly entropic issues are opened.

Recover the science frontier setup, and remark that, in the case of the geometric distribution, the entropy associated with a question raised at date $t = t_0$ is

$$H(t_0 + t, t_0) = \frac{-q(t_0 + t, t_0)\log_2[q(t_0 + t, t_0)] - [1 - q(t_0 + t, t_0)]\log_2[1 - q(t_0 + t, t_0)]}{1 - q(t_0 + t, t_0)} \quad (35)$$

The absence of any change in the level of attention attributed to each question ($\delta = 0$) signifies no movement on the science frontier, what is the same as saying that the probability of failure remains constant over time. In this circumstance, the level of entropy is immutable. The case of concentrated attention ($\delta > 0$) is synonymous of a decreasing probability of failure that, in the long-run, converges to zero; hence, when previously activated questions receive progressively more attention, entropy will fall from its initial value $H(t_0, t_0)$ towards zero. Finally, under dispersed attention ($\delta < 0$), as the science frontier of each research problem expands outwards, the probability of failure increases and so it does entropy. These results are the straightforward outcome of the direct relation between probability of failure and entropy, given the following derivative sign,

$$\frac{dH(t_0 + t, t_0)}{dq(t_0 + t, t_0)} = \frac{H(t_0 + t, t_0)}{1 - q(t_0 + t, t_0)} > 0 \quad (36)$$

The relation between attention assigned to a research question and the level of entropy becomes evident: more (less) attention associated to a particular scientific problem will imply a fall (a rise) in the respective entropy level. Because attributing more attention to a problem is an indication that scientists are more curious about it, we also establish, in this way, the highlighted link between curiosity and entropy.

More important than assessing the entropy of a single research question, is to evaluate the time trajectory of overall entropy, given the whole set of activated scientific challenges. Independently of the sign of δ, the total entropy at a time period $t = t_0 + T$ is, given the number of activated problems, (21),

$$\begin{aligned} H(t_0 + T) &= Q(t_0 + T) \times H(t_0 + T, t_0) \\ &= \frac{Q(t_0)}{(1 + \delta)^T} \times H(t_0 + T, t_0) \end{aligned} \quad (37)$$

In the long-term, given that $H(t_0 + T, t_0)$ converges to a constant value regardless of the trajectory of attention assigned to each problem, the growth-rate of entropy is straightforward to compute and it coincides with the growth rate of the number of answered questions as $T \to \infty$, i.e., $\gamma_H = -\delta/(1+\delta)$.

The above result, understood as sustained growth in entropy, is confined to the case $\delta < 0$. When $\delta = 0$, by the same formula, entropy converges to a constant long-term level, while if $\delta > 0$ we already know that $\lim_{T \to \infty} H(t_0 + T, t_0) = 0$ and, thus, no entropy change will occur in the long-run scenario.

Again, it is the dispersed attention case that is worth accentuating. By disseminating attention across a progressively larger amount of scientific problems, the research community feeds a never ending process of emergence of new questions. With the growth in the number of activated questions, overall entropy will also grow and curiosity will be as well an ever expanding attribute of the mission of science.

5. Extensions

5.1 Discrete Distributions of the Same Family

In this subsection, the science frontier analysis is expanded in order to allow for generalized forms of the geometric distribution. Namely, we will consider the exponentiated-exponential-geometric distribution (EEGD), approached in Alzaatreh *et al.* (2012) and the Kumaraswany-geometric distribution (KGD), discussed in Akinsete *et al.* (2014). They are both extensions of the geometric distribution and include this as a particular case. Given the cdf of each of the mentioned distributions, the respective science frontiers are, at the initial date:

$$EEGD : G(k, q(t_0, t_0)) = 1 - (1 - q(t_0, t_0)^k)^{\alpha},\ \alpha > 0 \tag{38}$$

$$KGD : G(k, q(t_0, t_0)) = [1 - (1 - q(t_0, t_0)^k)^{\alpha}]^{\beta},\ \alpha, \beta > 0 \tag{39}$$

Naturally, if $\alpha = \beta = 1$, in both cases one returns to the geometric distribution. Parameters α and β shape the form of the distribution but maintain the same underlying philosophy: a given probability of failure will determine how many questions will remain to be answered after *k* Bernoulli trials. Fig. 4 illustrates the position of the science frontier for each of the three distributions and for specific values of parameters. In particular, the probability of failure is set at $q(t_0, t_0) = 0.99$, and the values of the distributions' parameters are $\alpha = 0.5$, for the EEGD, and $\alpha = \beta = 2.5$, for the KGD. By shaping the location of the science frontier on the (k, G) referential, parameters α and β determine the success of the experimentation rounds and, therefore, they have a fundamental role concerning the efficiency of research. Note, for instance, taking into account the represented frontiers, that after 10 rounds the following figures are obtained: for the GD, approximately 9.56% of the posed problems have been successfully approached; this number increases to 30.92% in the case of the EEGD distribution, and falls to 0.71% for the KGD. Consequently, although built under a same general structure, the three distributions tell considerably different stories about the efficiency of scientific research.

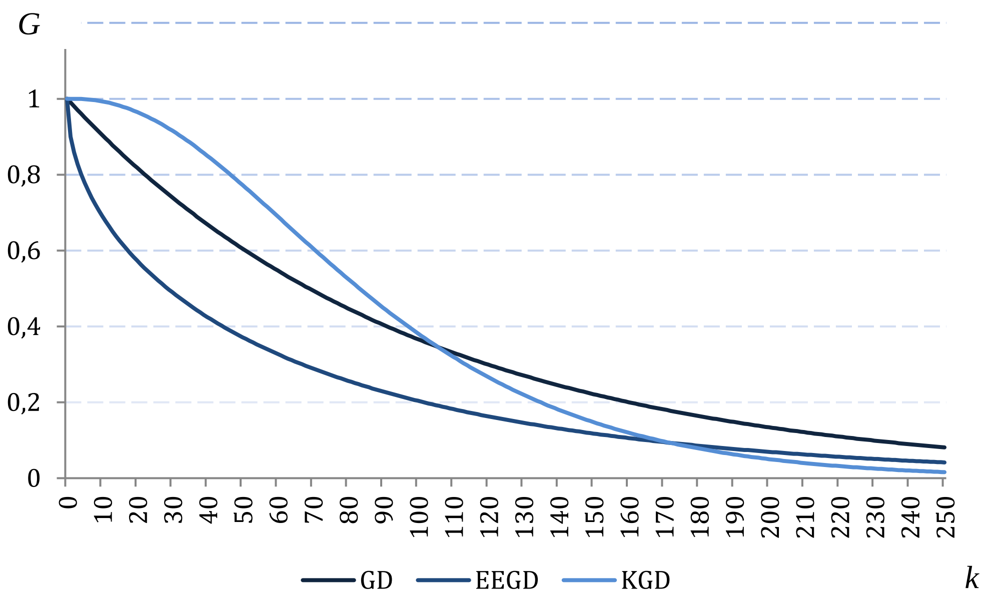

Fig.4. Science frontier, for different geometric distributions

($q(t_0, t_0) = 0.99$; *GD:* $\alpha = \beta = 1$; *EEGD:* $\alpha = 0.5, \beta = 1$; *KGD:* $\alpha = \beta = 2.5$).

Regardless of the different positions of the science frontier in the above examples, the dynamics implied by the evolution of the attention attributed to each research question do not depart from what was stated in previous sections concerning the geometric distribution. Note, in the first place, that the case of unchangeable attention, $\delta = 0$, continues to determine, for all positive α and β, that the science frontier will suffer no displacement over time. In the most general KGD form, the difference equation for the change in the distribution is such that

$$G(k, q(t_0 + t + 1, t_0)) = \left[1 - \left(1 - [q(t_0 + t, t_0) \times q(t_0 + t, t_0)^{\Delta w(t_0+t, t_0)}]^k\right)^{\alpha}\right]^{\beta} \quad (40)$$

Under $\delta = 0$,

$$G(k, q(t_0 + t + 1, t_0)) = \left[1 - \left(1 - (q(t_0 + t, t_0))^k\right)^{\alpha}\right]^{\beta} = G(k, q(t_0 + t, t_0)) \quad (41)$$

and so the frontier does not change position from one period to the next; again, the memoryless property is present: independently of the number of Bernoulli trials undertaken in the precedent time period, the distribution maintains its shape in the current period.

Changes in attention, both in the direction of concentration or dispersion, will move the frontier. The qualitative results are those already known: allocating additional attention to each question will pull the frontier inwards, as the probability of failure falls to zero; a steady-state $G^* = 0$ is necessarily reached; if the attention is dispersed, the frontier moves outwards until reaching some long-term locus such that $G(k, q(t_0, t_0)) < G^* < 1$. Long-run results, regarding the growth rate of correctly answered science puzzles, will be, given the characterized motion of the science frontier, of a same qualitative nature of those found for the geometric distribution. In fact, changing the shape of the distribution will affect transitional dynamics but it will be innocuous in what respects the BGP result. Fig. 5 displays the trajectories followed by the growth rate of answered questions for the three different distributions, assuming the endogenous growth case of dispersed attention. Parameter values employed to represent the *GD*, *EEGD* and *KGD* distributions are the same as those used to draw the frontiers in Fig. 4. Although disparities between trajectories are evident in the transient phase, they are dissipated over time.

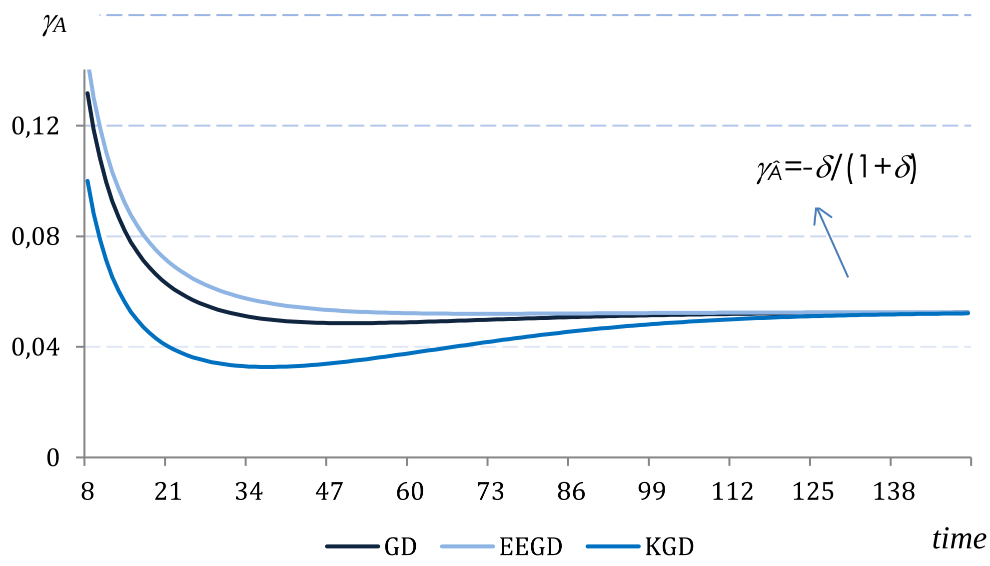

Fig. 5. Trajectories of the growth rate of answered questions - geometric, EEGD and KGD distributions

$$(\delta = -0.05;\ k = 5;\ w(t_0, t_0) = 1;\ q(t_0, t_0) = 0.99).$$

5.2 Continuous-Time

The science dynamics framework is, in what follows, adapted to a continuous-time scenario. Bernoulli trials and the geometric distribution continue to characterize, as before, the process of scientific creation in a particular time period *t*, but in this new scenario time flows continuously. Consider, once more, rate $\delta \in \mathbb{R}$, and now take time intervals of a given length *h*, such that

$$w(t+h,t_0) - w(t,t_0) = \delta h w(t,t_0) \tag{42}$$

The science frontier will evolve, over time, as in (4),

$$G(k,q(t+h,t_0)) = G(k,q(t,t_0))^{1+\delta h w(t,t_0)} \tag{43}$$

The corresponding rate of change is

$$\frac{G(k,q(t+h,t_0)) - G(k,q(t,t_0))}{G(k,q(t,t_0))h} = \frac{G(k,q(t,t_0))^{\delta h w(t,t_0)} - 1}{h} \tag{44}$$

Taking the definition of derivative, one transforms (44) into an ordinary differential equation (ODE), by noting that

$$\frac{\partial[G(k,q(t,t_0))]}{\partial t} \times \frac{1}{G(k,q(t,t_0))} = \lim_{h \to 0} \frac{G(k,q(t,t_0))^{\delta h w(t,t_0)} - 1}{h} \tag{45}$$

an expression that is equivalent to

$$[\ln G(k, \dot{q}(t,t_0))] = \delta w(t,t_0)[\ln G(k,q(t,t_0))] \tag{46}$$

The growth rate of attention is, in the continuous-time framework,

$$\frac{\dot{w}(t,t_0)}{w(t,t_0)} = \delta \tag{47}$$

which has, as solution,

$$w(t,t_0) = w(t_0,t_0) \times \exp(\delta t) \tag{48}$$

Replacing the value of attention, given by (48), into the research frontier ODE, (46), it is straightforward to compute the solution of the differential equation, which will be given by the following expression,

$$\ln G(k,q(t,t_0)) = \ln G(k,q(t_0,t_0)) \times \exp\{w(t_0,t_0) \times [\exp(\delta t) - 1]\} \tag{49}$$

or, suppressing the logs,

$$G(k,q(t,t_0)) = G(k,q(t_0,t_0))^{\exp\{w(t_0,t_0)\times[\exp(\delta t)-1]\}} \tag{50}$$

Solution (50) furnishes important guidance concerning the dynamics of the science frontier. Observe that if $\delta > 0$, then $G(k,q(t,t_0))$ converges to zero: $G^* = 0$; if $\delta = 0$, then $G(k,q(t,t_0)) = G(k,q(t_0,t_0))$ in the long-term, i.e., $G^* = G(k,q(t_0,t_0))$; and if $\delta < 0$, then $\lim_{t\to\infty} G(k,q(t,t_0)) = G(k,q(t_0,t_0))^{\exp[-w(t_0,t_0)]}$; because $0 < \exp[-w(t_0,t_0)] < 1$, the long-term science frontier, G^*, will be such that $G(k,q(t_0,t_0)) < G^* < 1$. Comparing these results with the ones obtained in discrete time, it is straightforward to encounter the respective similarities. Fig. 6 displays a phase diagram that characterizes the evolution of the science frontier, given the motion of attention. Starting from an initial point $(w(t_0,t_0);G(k,q(t_0,t_0)))$, three possibilities are feasible: *i*) attention remains constant over time, and so does the science frontier; *ii*) attention grows positively over time, implying that the frontier converges to zero; *iii*) attention attributed to each individual question falls over time, which means a convergence to a given frontier position G^*, which we find somewhere above the initial frontier location.

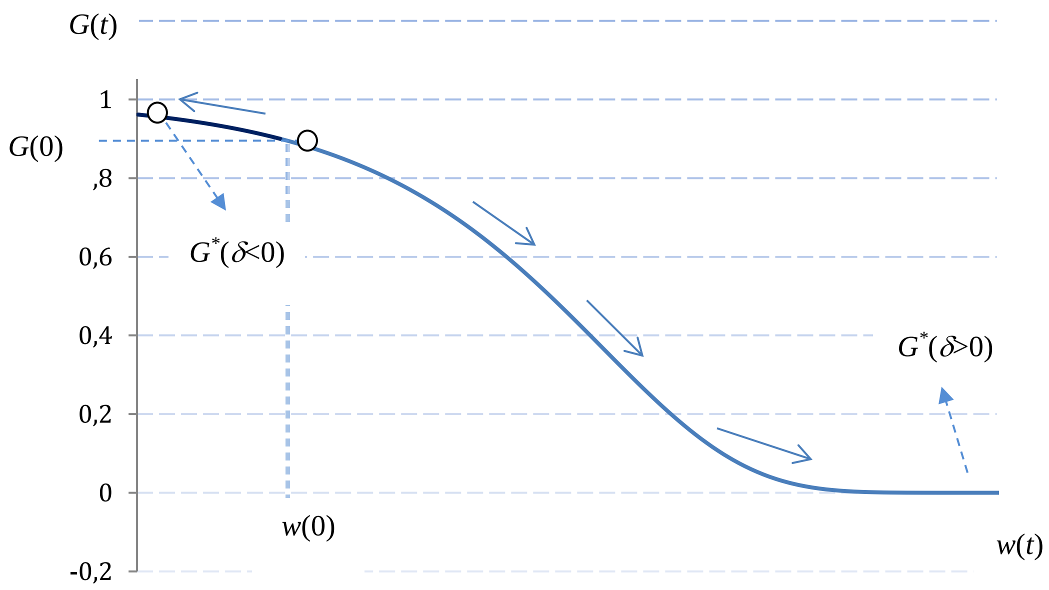

Fig.6. Phase diagram - dynamics of the science frontier.

Also as in the discrete-time case, despite the different positions of the science frontier in the BGP given the value of δ, the long-term scenario will be such that all questions of the first vintage have been answered. The number of questions that remain to be answered at $t = t_0 + T$ is

$$Q(t) = Q(t_0) \times G(k, q(t_0, t_0))^{\int_0^T \exp\{w(t_0,t_0)\times[\exp(\delta t)-1]\}dt} \quad (51)$$

Regarding that

$$\lim_{T\to\infty} \int_0^T \exp\{w(t_0, t_0) \times [\exp(\delta t) - 1]\}dt = \infty, \forall \delta \in \mathbb{R} \quad (52)$$

one concludes that the number of unanswered questions falls to zero, and therefore all questions end up with an associated successful answer.

An analogous procedure of analysis to the one adopted in the discrete-time version of the model conducts to the following aggregate values:

i) Number of questions to answer at moment $t = t_0 + T$,

$$Q(t) = \frac{Q(t_0)}{\exp(\delta T)} \tag{53}$$

ii) Number of questions solved at time $t = t_0 + T$,

$$A(t) = \frac{Q(t_0)}{\exp(\delta T)} \times [1 - G(k, q(t_0 + T, t_0))] \tag{54}$$

iii) Accumulated valued of solved problems at date $t = t_0 + T$,

$$\widehat{A}(t) = Q(t_0) \times \int_0^T \frac{1 - G(k, q(t_0 + t, t_0))}{\exp(\delta t)} dt \tag{55}$$

From (55), it is straightforward to arrive to the result in proposition 4.

Proposition 4. *The long-term growth-rate of the stock of successfully met scientific challenges is, in the continuous-time setting and assuming dispersed attention,* $\gamma_{\widehat{A}} = -\delta$
.

Proof See appendix ■

The growth rate results in discrete-time and continuous-time are not exactly the same. In continuous-time, the BGP is characterized by a growth rate that coincides with the absolute value of the rate of change of attention assigned to scientific problems over time. While in discrete-time the growth rate is $\gamma_{\widehat{A}} = -\frac{\Delta w(t_0+T, t_0)}{w(t_0+T+1, t_0)}$, the continuous-time analogue is, more plainly, $\gamma_{\widehat{A}} = \frac{\dot{w}(t, t_0)}{w(t, t_0)}$.

In parallel with the discrete-time case, the analysis can be extended to include knowledge obsolescence, attention growth and increasing value of innovations. Recovering the obsolescence rate $\sigma \in (0, 1)$, the accumulated value of solved problems has correspondence in the following expression,

$$\begin{aligned}\hat{V}(t) &= Q(t_0) \times \int_0^T \exp[-\sigma(T-t)] \frac{1 - G(k, q(t_0 + t, t_0))}{\exp(\delta t)} dt \\ &= Q(t_0) \times \int_0^T \frac{1 - G(k, q(t_0 + t, t_0))}{\exp[(\delta - \sigma)t + \sigma T]} dt \end{aligned} \tag{56}$$

For the cases where a positive growth rate of attention, $\gamma > 0$, or an increasing vintage value, $v > 0$, are assumed, the accumulated levels of solved problems are, respectively,

$$\hat{V}(t) = Q(t_0) \times \int_0^T \exp(\gamma t) \times \frac{1 - G(k, q(t_0 + t, t_0))}{\exp(\delta t)} dt \tag{57}$$

and

$$\hat{V}(t) = Q(t_0) \times \int_0^T \exp(v t) \times \frac{1 - G(k, q(t_0 + t, t_0))}{\exp(\delta t)} dt \tag{58}$$

Under a same logic as the one adopted in the proof of proposition 4 to derive the growth rate of the sum of all successfully answered questions, it is straightforward to calculate the long-term growth rates for the modified structure of analysis. Again, the results display some similarities with discrete-time,

i) Growth rate with knowledge obsolescence,

$$\frac{\dot{\hat{V}}(t)}{\hat{V}(t)} = -\delta \tag{59}$$

ii) Growth rate with a varying stock of attention,

$$\frac{\dot{\hat{V}}(t)}{\hat{V}(t)} = \gamma - \delta \tag{60}$$

iii) Growth rate with increasing value of new vintages,

$$\frac{\dot{\hat{V}}(t)}{\hat{V}(t)} = v - \delta \tag{61}$$

iv) Growth rate with a varying stock of attention and increasing value of new vintages,

$$\frac{\dot{\hat{V}}(t)}{\hat{V}(t)} = \gamma + v - \delta \tag{62}$$

6. Conclusion: Scientific Curiosity as Dispersed Attention

Scientific curiosity knows no limits. Whenever a research question is successfully answered, a burst of new interrogations is likely to come to the surface and occupy the minds of human beings. Attention to scientific challenges may grow, but attention is, in fact, a scarce resource. Therefore, the doubt must be raised: how can one conciliate ever increasing unbounded curiosity with a constrained attention endowment? The paper furnishes a candidate explanation for this apparent paradox: by assuming a science frontier, constructed over a process of trial-and-error, it is straightforward to demonstrate that spreading attention over additional research challenges is all that is required for a result of sustained long-term growth to thrive. No increasing value of new science vintages or strong obsolescence of previous discoveries are essential in this process. Growth occurs just because of the dispersion over time of attention to new scientific objects.

Two central ideas are worth highlighting from the undertaken analysis. On one hand, it served to contextualize curiosity in the reflection about the evolution of research and science. Although the notion of curiosity might be subject to various possible interpretations, in science it can be associated with the progressively wider interest in all conceivable objects in nature, society and abstract thinking. This increasing interest in everything that surround us is by itself, as discussed, an engine of everlasting science evolution; one does not need to allocate progressively more resources to science to observe sustained growth in scientific knowledge. On the other hand, the advanced arguments bring, as well, a significant result for economic growth theory, a result that is visible without ambiguities through the observation of the trajectories in Fig. 3: a same framework of analysis, considering a fixed attention input, is able to replicate, for different values of a single parameter, both the neoclassical and the endogenous growth outcomes; endogenous growth requires dispersed attention, while zero long-term growth is the outcome of concentrating attention or, at least, maintaining it at the starting point.

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Appendix

Proof of proposition 1

The steady-state result, (9) under $t\to\infty$, might be scrutinized for each of the possibilities concerning the sign of parameter δ. Define

$$G^* \equiv \lim_{T\to\infty} G(k,q(t_0+T,t_0)) \tag{63}$$

In case $\delta = 0$, one immediately observes that $a_\tau = 1$; hence,

$$\lim_{T\to\infty}\left\{\exp\left[\sum_{\tau=0}^{T-1}\ln(a_\tau)\right]\right\} = \exp(0) = 1 \tag{64}$$

and, therefore, $G^* = G(k,q(t_0,t_0))$: no attention change signifies no movement of the science frontier (it perpetuates itself on its initial position).

If $\delta > 0$, then a_τ explodes to infinity and

$$\lim_{T\to\infty}\left\{\exp\left[\sum_{\tau=0}^{T-1}\ln(a_\tau)\right]\right\} = \exp(\infty) = \infty \tag{65}$$

implying that $G^* = 0$, i.e., growing attention drives the system to a long-term state where the science frontier coincides with the origin in the (k, G) referential.

The third possibility, $\delta < 0$, requires a finer examination. In this circumstance, observe the following about the evolution of term a_τ,

$$a_0 = 1 + \delta w(t_0, t_0) < 1 \tag{66}$$

$$\lim_{\tau\to\infty}(a_\tau) = 1 \tag{67}$$

$$\frac{\partial a_\tau}{\partial \tau} = \delta(1+\delta)^\tau \ln(1+\delta) w(t_0, t_0) > 0 \tag{68}$$

Hence, as τ goes to infinity, a_τ follows an increasing trajectory that converges to 1. Note that one must guarantee that a_τ is positive, what implies imposing condition

$$w(t_0, t_0) < -\frac{1}{\delta} \tag{69}$$

For $0 < a_\tau < 1$, it is true that $\ln(a_\tau) < 0, \forall\tau$, and that $\lim_{\tau\to\infty}[\ln(a_\tau)] = 0$. Also, as a consequence,

$$\sum_{\tau=0}^{\infty}\ln(a_\tau) < 0 \tag{70}$$

This last condition may imply one of two outcomes: the series might diverge to -∞ or absolutely converge to a finite negative value. Convergence holds if the condition imposed by the series ratio test is satisfied, i.e., the series $\sum_{\tau=0}^{\infty} \ln(a_\tau)$ converges if inequality $L < 1$ holds, with $L \equiv \lim_{\tau\to\infty} \left| \frac{\ln(a_{\tau+1})}{\ln(a_\tau)} \right|$.

Applying the L'Hopital rule,

$$\lim_{\tau\to\infty} \left| \frac{\ln(a_{\tau+1})}{\ln(a_\tau)} \right| = \lim_{\tau\to\infty} \left| \frac{\frac{\partial \ln(a_{\tau+1})}{\partial\tau}}{\frac{\partial \ln(a_\tau)}{\partial\tau}} \right| = \lim_{\tau\to\infty} \left| (1+\delta)\frac{a_\tau}{a_{\tau+1}} \right| = 1+\delta \tag{71}$$

Given that $\delta < 0$, the limit is a lower than 1 value, i.e., $L < 1$, what proves convergence. Hence,

$$\sum_{\tau=0}^{\infty} \ln(a_\tau) = \varsigma(\delta; w(t_0, t_0)) \tag{72}$$

with $\varsigma(\delta; w(t_0, t_0))$ a negative finite constant, for given values of the attention growth rate and of the initial level of attention. Consequently,

$$\lim_{T\to\infty} \left\{ \exp\left[\sum_{\tau=0}^{T-1} \ln(a_\tau) \right] \right\} = \exp[\varsigma(\delta; w(t_0, t_0))] < 1 \tag{73}$$

Therefore, in the case $\delta < 0$ we have $G(k, q(t_0, t_0)) < G^* < 1$, that is, whenever attention falls over time the share of unanswered questions, from those that remain activated, is higher in the long-run than in the initial state.

Despite the diversity of long-term outcomes concerning the science frontier for the different signs of δ, in all the three cases there will not remain any scientific problems to be solved in the steady-state. This is straightforward to confirm by noticing that for some $t = t_0 + T > t_0$, the number of questions raised at $t = t_0$ yet to be addressed is the amount

$$\begin{aligned} Q(t) &= Q(t_0) \times G(k, q(t_0, t_0)) \times G(k, q(t_0+1, t_0)) \times \ldots \times G(k, q(t_0+T, t_0)) \\ &= Q(t_0) \times G(k, q(t_0, t_0))^{\left(1+\sum_{\zeta=0}^{T-1}\left\{\exp\left[\sum_{\tau=0}^{\zeta} \ln(a_\tau)\right]\right\}\right)} \end{aligned} \tag{74}$$

Independently of the sign of the growth rate of attention parameter δ, observe that

$$\lim_{T\to\infty}\left(1+\sum_{\zeta=0}^{T-1}\left\{\exp\left[\sum_{\tau=0}^{\zeta}\ln(a_\tau)\right]\right\}\right)=\infty \qquad (75)$$

This is a straightforward outcome for $\delta > 0$. For $\delta = 0$, the limit expression reduces to $\lim_{T\to\infty}\sum_{\zeta=0}^{T-1}(1)=\lim_{T\to\infty} T=\infty$, and for $\delta < 1$, the result is similar, since

$$\lim_{T\to\infty}\sum_{\zeta=0}^{T-1}\{\exp[\varsigma(\delta;w(t_0,t_0))]\}=\lim_{T\to\infty} T\times\{\exp[\varsigma(\delta;w(t_0,t_0))]\}=\infty \qquad (76)$$

Note that $\exp[\varsigma(\delta;w(t_0,t_0))]$ is some constant in the interval (0,1). The zero unsolved questions at the steady-state result is, in this way, confirmed, since

$$Q(t)=Q(t_0)\times[G(k,q(t_0,t_0))]^{\infty}=0 \qquad (77)$$

Proof of proposition 2

Regardless of the sign of δ, the stock of knowledge at date $t = t_0 + T$, measured by the accumulated number of successfully answered research questions, is given by expression (23). A BGP for the evolution of scientific knowledge corresponds to the trajectory followed by $\widehat{A}(t_0+T)$, as T tends to infinity. Therefore, to know the nature of the BGP one needs to compute the growth rate of $\widehat{A}$ between two consecutive periods, T and T+1. This is

$$\gamma_{\hat{A}} = \frac{\sum_{t=0}^{T+1} \frac{1-G(k,q(t_0+t,t_0))}{(1+\delta)^t} - \sum_{t=0}^{T} \frac{1-G(k,q(t_0+t,t_0))}{(1+\delta)^t}}{\sum_{t=0}^{T} \frac{1-G(k,q(t_0+t,t_0))}{(1+\delta)^t}} \tag{78}$$

Equivalently,

$$\gamma_{\hat{A}} = \frac{\frac{1-G(k,q(t_0+T+1,t_0))}{(1+\delta)^{T+1}}}{\sum_{t=0}^{T} \frac{1-G(k,q(t_0+t,t_0))}{(1+\delta)^t}} \tag{79}$$

The above expression might be rearranged,

$$\gamma_{\hat{A}} = \frac{1}{\sum_{t=0}^{T} \left[\frac{1-G(k,q(t_0+t,t_0))}{1-G(k,q(t_0+T+1,t_0))} \times (1+\delta)^{T+1-t} \right]} \tag{80}$$

and, finally, written as

$$\gamma_{\hat{A}} = \frac{1}{1+\delta} \frac{1}{\sum_{t=0}^{T} \left[\frac{1-G(k,q(t_0+t,t_0))}{1-G(k,q(t_0+T+1,t_0))} \times (1+\delta)^{T-t} \right]} \tag{81}$$

To proceed, one needs to interpret the sum in the denominator of the expression. This sum can be separated into two terms,

$$\sum_{t=0}^{T} \left[\frac{1-G(k,q(t_0+t,t_0))}{1-G(k,q(t_0+T+1,t_0))} \times (1+\delta)^{T-t} \right] =$$

$$\sum_{t=0}^{t'-1} \left[\frac{1-G(k,q(t_0+t,t_0))}{1-G(k,q(t_0+t',t_0))} \times (1+\delta)^{T-t} \right] + \sum_{t=t'}^{T} \left[\frac{1-G(k,q(t_0+t,t_0))}{1-G(k,q(t_0+T+1,t_0))} \times (1+\delta)^{T-t} \right]$$

(82)

where *t'* corresponds to the point where, asymptotically, the transient phase of the evolution of *G* is complete and the BGP is attained. In this case, for an infinite *T* and a finite *t'*, the value $(1+\delta)^{T-t}$ will be approximately zero, for any value of *t*, such that

$$\sum_{t=0}^{t'-1}\left[\frac{1-G(k,q(t_0+t,t_0))}{1-G(k,q(t_0+t',t_0))}\times(1+\delta)^{T-t}\right]\simeq 0 \tag{83}$$

Next, recall, from the proof of proposition 1, that the science frontier converges to a long-term constant value that depends on the growth rate of attention:

$$\begin{cases}\delta>0: G^*=0\\ \delta=0: G^*=G(k,q(t_0+t,t_0)),\forall t=0,1,\dots\\ \delta<0: G^*=G(k,q(t_0,t_0))^{\exp[\varsigma(\delta;w(t_0,t_0))]}\in(G(k,q(t_0+t,t_0)),1)\end{cases} \tag{84}$$

Thus, concerning the second term of the above equality,

$$\begin{aligned}\sum_{t=t'}^{T}\left[\frac{1-G(k,q(t_0+t,t_0))}{1-G(k,q(t_0+T+1,t_0))}\times(1+\delta)^{T-t}\right] &=\sum_{t=t'}^{T}\left[\frac{1-G^*}{1-G^*}\times(1+\delta)^{T-t}\right]\\ &=\sum_{t=t'}^{T}(1+\delta)^{T-t}=\sum_{t=0}^{T-t'}(1+\delta)^{t}\end{aligned} \tag{85}$$

The value of the geometric series will depend on the sign of δ. For $\delta=0$,

$$\sum_{t=0}^{T-t'}(1+\delta)^{t}=T-t'+1 \tag{86}$$

and, therefore,

$$\gamma_{\hat{A}}=\frac{1}{T-t'+1} \tag{87}$$

As $T\to\infty$, the growth rate will converge to zero.

For $\delta > 0$, it is straightforward to notice that

$$\lim_{T\to\infty}\sum_{t=0}^{T-t'} (1+\delta)^t = \infty \tag{88}$$

what implies a long-term zero growth rate for scientific knowledge, also in this case.

Finally, let us consider $\delta < 0$. Now,

$$\sum_{t=0}^{T-t'} (1+\delta)^t = \frac{1-(1+\delta)^{T-t'+1}}{-\delta} \tag{89}$$

and

$$\lim_{T\to\infty}\sum_{t=0}^{T-t'} (1+\delta)^t = -\frac{1}{\delta} \tag{90}$$

Replacing this value into the growth rate,

$$\gamma_{\hat{A}} = \frac{1}{1+\delta}\frac{1}{-\frac{1}{\delta}} = -\frac{\delta}{1+\delta} \tag{91}$$

Under dispersed attention, the BGP is characterized by a positive growth rate, that depends solely on the decreasing rate of attention given to approach scientific challenges over time.

Proof of proposition 3

The computation of the growth rate of accumulated knowledge with the new elements present in section 4.2 follows the same steps as the calculus of the growth rate for the original accumulated amount of scientific answers. Let us start by considering, as additions to the benchmark model, solely the presence of knowledge obsolescence. The growth rate of $\hat{V}(t)$ as defined in (26) is

$$\gamma_{\widehat{V}} = \frac{Q(t_0) \times \sum_{t=0}^{T+1} \left[(1-\sigma)^{T+1-t} \times \frac{1-G(k,q(t_0+t,t_0))}{(1+\delta)^t} \right] - Q(t_0) \times \sum_{t=0}^{T} \left[(1-\sigma)^{T-t} \times \frac{1-G(k,q(t_0+t,t_0))}{(1+\delta)^t} \right]}{Q(t_0) \times \sum_{t=0}^{T} \left[(1-\sigma)^{T-t} \times \frac{1-G(k,q(t_0+t,t_0))}{(1+\delta)^t} \right]} \tag{92}$$

This expression is equivalent to

$$\begin{aligned}
\gamma_{\widehat{V}} &= \frac{\frac{1-G(k,q(t_0+T+1,t_0))}{(1+\delta)^{T+1}} - \sigma \times \sum_{t=0}^{T} \left[(1-\sigma)^{T-t} \times \frac{1-G(k,q(t_0+t,t_0))}{(1+\delta)^t} \right]}{\sum_{t=0}^{T} \left[(1-\sigma)^{T-t} \times \frac{1-G(k,q(t_0+t,t_0))}{(1+\delta)^t} \right]} \\
&= \frac{\frac{1-G(k,q(t_0+T+1,t_0))}{(1+\delta)^{T+1}}}{\sum_{t=0}^{T} \left[(1-\sigma)^{T-t} \times \frac{1-G(k,q(t_0+t,t_0))}{(1+\delta)^t} \right]} - \sigma \\
&= \frac{1}{1+\delta} \frac{1}{\sum_{t=0}^{T} \left\{ \frac{1-G(k,q(t_0+t,t_0))}{1-G(k,q(t_0+T+1,t_0))} \times [(1-\sigma)(1+\delta)]^{T-t} \right\}} - \sigma
\end{aligned} \tag{93}$$

As in the proof of proposition 2, we separate the sum in the denominator into two terms, thus distinguishing, asymptotically, between the transient phase and the long-term locus,

$$\begin{aligned}
&\sum_{t=0}^{T} \left\{ \frac{1-G(k,q(t_0+t,t_0))}{1-G(k,q(t_0+T+1,t_0))} \times [(1-\sigma)(1+\delta)]^{T-t} \right\} \\
&= \sum_{t=0}^{t'-1} \left\{ \frac{1-G(k,q(t_0+t,t_0))}{1-G(k,q(t_0+T+1,t_0))} \times [(1-\sigma)(1+\delta)]^{T-t} \right\} \\
&+ \sum_{t=t'}^{T} \left\{ \frac{1-G(k,q(t_0+t,t_0))}{1-G(k,q(t_0+T+1,t_0))} \times [(1-\sigma)(1+\delta)]^{T-t} \right\}
\end{aligned} \tag{94}$$

Suppressing the first term in the sum, and taking $\frac{1-G(k,q(t_0+t,t_0))}{1-G(k,q(t_0+T+1,t_0))} = 1$ for any $t \geq t'$, the expression simplifies to

$$\sum_{t=t'}^{T} [(1-\sigma)(1+\delta)]^{T-t} = \sum_{t=0}^{T-t'} [(1-\sigma)(1+\delta)]^{t} = \frac{1-[(1-\sigma)(1+\delta)]^{T-t'+1}}{1-(1-\sigma)(1+\delta)} \tag{95}$$

The infinite horizon evaluation conducts to the outcome

$$\lim_{T\to\infty} \sum_{t=0}^{T-t'} (1+\delta)^{t} = \frac{1}{1-(1-\sigma)(1+\delta)} \tag{96}$$

Replacing in the growth rate expression,

$$\gamma_{\widehat{V}} = \frac{1}{1+\delta}\frac{1}{\frac{1}{1-(1-\sigma)(1+\delta)}} - \sigma = -\frac{\delta}{1+\delta} \tag{97}$$

The introduction of obsolescence, on its own, does not modify the original endogenous growth result.

To obtain the long-run growth rate for the other two specifications of the model, a similar procedure must be adopted. The growth rates requiring evaluation are, respectively,

$$\gamma_{\widehat{V}} = \frac{Q(t_0) \times \sum_{t=0}^{T+1} \left\{ \left(\frac{1+\gamma}{1+\delta}\right)^{t} \times [1-G(k,q(t_0+t,t_0))] \right\} - Q(t_0) \times \left\{ \left(\frac{1+\gamma}{1+\delta}\right)^{t} \times [1-G(k,q(t_0+t,t_0))] \right\}}{Q(t_0) \times \sum_{t=0}^{T} \left\{ \left(\frac{1+\gamma}{1+\delta}\right)^{t} \times [1-G(k,q(t_0+t,t_0))] \right\}} \tag{98}$$

and

$$\gamma_{\hat{V}} = \frac{Q(t_0) \times \sum_{t=0}^{T+1} \left\{ \left(\frac{1+v}{1+\delta}\right)^t \times [1 - G(k, q(t_0 + t, t_0))] \right\} - Q(t_0) \times \left\{ \left(\frac{1+v}{1+\delta}\right)^t \times [1 - G(k, q(t_0 + t, t_0))] \right\}}{Q(t_0) \times \sum_{t=0}^{T} \left\{ \left(\frac{1+v}{1+\delta}\right)^t \times [1 - G(k, q(t_0 + t, t_0))] \right\}} \quad (99)$$

Taking exactly the same steps, and noticing that

$$\lim_{T\to\infty} \sum_{t=0}^{T-t'} \left(\frac{1+\delta}{1+\gamma}\right)^t = \frac{1}{1 - \left(\frac{1+\delta}{1+\gamma}\right)} = \frac{1+\gamma}{\gamma - \delta} \quad (100)$$

$$\lim_{T\to\infty} \sum_{t=0}^{T-t'} \left(\frac{1+\delta}{1+v}\right)^t = \frac{1}{1 - \left(\frac{1+\delta}{1+v}\right)} = \frac{1+v}{\gamma - \delta} \quad (101)$$

the respective long-term growth rates are

$$\gamma_{\hat{V}} = \frac{1+\gamma}{1+\delta} \times \frac{1}{\frac{1+\gamma}{\gamma-\delta}} = \frac{\gamma - \delta}{1+\delta} \quad (102)$$

$$\gamma_{\hat{V}} = \frac{1+v}{1+\delta} \times \frac{1}{\frac{1+v}{\gamma-\delta}} = \frac{v - \delta}{1+\delta} \quad (103)$$

Attention growth and increasing vintage value can be combined. The corresponding BGP rate is

$$\gamma_{\hat{V}} = \frac{(1+\gamma)(1+v)}{1+\delta} \times \frac{1}{\frac{(1+\gamma)(1+v)}{(1+\gamma)(1+v)-(1+\delta)}} = \frac{(1+\gamma)(1+v)}{1+\delta} - 1 \quad (104)$$

Proof of proposition 4

Let us begin by addressing the growth of $A(t)$ and then proceed to the corresponding cumulative value.

The derivative with respect to time of $A(t)$ is

$$\begin{aligned}\dot{A}(t) &= -Q(t_0)\times\frac{\dot{G}(k,q(t_0+t,t_0)) + \delta[1-G(k,q(t,t_0))]}{\exp(\delta t)}\\ &= -Q(t_0)\times\frac{\delta w(t_0,t_0)\exp(\delta t)G(k,q(t,t_0))\ln[G(k,q(t,t_0))]+\delta[1-G(k,q(t,t_0))]}{\exp(\delta t)}\\ &= -\delta Q(t_0)\times\frac{G(k,q(t,t_0))[w(t_0,t_0)\exp(\delta t)\ln[G(k,q(t,t_0))]-1]+1}{\exp(\delta t)}\end{aligned} \quad (105)$$

Dividing by the expression of $A(t)$,

$$\begin{aligned}\frac{\dot{A}(t)}{A(t)} &= \frac{-\delta Q(t_0)\times\frac{G(k,q(t,t_0))[w(t_0,t_0)\exp(\delta t)\ln[G(k,q(t,t_0))]-1]+1}{\exp(\delta t)}}{\frac{Q(t_0)}{\exp(\delta t)}\times[1-G(k,q(t,t_0))]}\\ &= -\delta\times\frac{G(k,q(t,t_0))[w(t_0,t_0)\exp(\delta t)\ln[G(k,q(t,t_0))]-1]+1}{1-G(k,q(t,t_0))}\end{aligned} \quad (106)$$

For $t\to\infty$,

$$\frac{\dot{A}(t)}{A(t)} = -\delta\times\frac{G(k,q(t,t_0))\times(0-1)+1}{1-G(k,q(t,t_0))} \Leftrightarrow \frac{\dot{A}(t)}{A(t)} = -\delta \quad (107)$$

If, alternatively, one takes $\widehat{A}(t)$, then the respective change value is

$$\dot{\widehat{A}}(t) = -\delta Q(t_0)\times\int_0^t\frac{G(k,q(t,t_0))[w(t_0,t_0)\exp(\delta t)\ln[G(k,q(t,t_0))]-1]+1}{\exp(\delta t)}dt \quad (108)$$

and the growth rate comes

$$\frac{\dot{\hat{A}}(t)}{\hat{A}(t)} = \frac{-\delta Q(t_0) \times \int_0^T \frac{G(k,q(t,t_0))[w(t_0,t_0)\exp(\delta t)\ln[G(k,q(t,t_0))]-1]+1}{\exp(\delta t)} dt}{Q(t_0) \times \int_0^t \frac{1-G(k,q(t_0+t,t_0))}{\exp(\delta t)} dt} \quad (109)$$

For $T \to \infty$,

$$\frac{\dot{\hat{A}}(t)}{\hat{A}(t)} = \frac{-\delta Q(t_0) \times \int_0^T \frac{G(k,q(t,t_0))\times(0-1)+1}{\exp(\delta t)} dt}{Q(t_0) \times \int_0^T \frac{1-G(k,q(t_0+t,t_0))}{\exp(\delta t)} dt} \Leftrightarrow \frac{\dot{\hat{A}}(t)}{\hat{A}(t)} = -\delta \quad (110)$$

This result is valid for $\delta < 0$. Under $\delta \geq 0$, observe that $\dot{A}(t) = 0$ and, thus, the BGP encloses a result of zero growth.

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